INDIAN STATISTICAL INSTITUTE, Bangalore Centre
Endsem
April 24, 2026

Course Name: Complex Analysis Duration: 2 hours

Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

1. State whether the following statements are TRUE/ FALSE with proper justification.

   (a) If f is a one-one analytic function in the open unit disk $\mathbb{D}$, then f' must be non-vanishing in $\mathbb{D}$. [2]

   (b) The quantity $\{|f'(0)| : f : \mathbb{D} \to \mathbb{D} \text{ analytic}\}$ need not be bounded above. [3]

   (c) Let f be a non-constant entire function, then for every positive real number $r > 0$, the closure of $\{z : |f(z)| < r\}$ is equal to $\{z : |f(z)| \leq r\}$. [3]

   (d) Let f be analytic in a region containing $\mathbb{D}$ and satisfying $|f(z)| < 1$ whenever $|z| \leq 1$. Then f has a fixed point a with $|a| < 1$. [2]

2. (a) Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function. Let $\alpha \in \mathbb{D}$ and $f(\alpha) = 0$. Prove that $|f(0)| \leq |\alpha|$.

   (b) Let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function such that $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. Let $f(a) = 0 = f(-a)$ for some $a \in \mathbb{D}$. Then prove that $|f(0)| \leq |a|^2$.

   [3+7]

3. (a) Give the definition of singularity at infinity. Also give examples of all possible singularities at infinity.

   (b) Prove that the automorphism group of $\mathbb{C}$ is

$$\mathrm{Aut}(\mathbb{C}) = \{\mathrm{az} + \mathrm{b} : 0 \neq \mathrm{a} \in \mathbb{C}, \mathrm{b} \in \mathbb{C}\}.$$

   .

   [3+7]

4. (a) Let f be analytic in $\mathbb{C}$ except finitely many singularities $p_1, p_2, \ldots, p_m$. Then show that residue of f at ∞

$$\mathrm{Res}(f, \infty) = -\sum_{k=1}^{m} \mathrm{Res}(f, p_k).$$

   (b) Calculate $\int_0^\infty \frac{x^2 dx}{x^4 + x^2 + 1}$.

   [3+7]